

On the game domination number of graphs with given minimum degree*

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Abstract

In the domination game, introduced by Brešar, Klavžar and Rall in 2010, Dominator and Staller alternately select a vertex of a graph G . A move is legal if the selected vertex v dominates at least one new vertex – that is, if we have a $u \in N[v]$ for which no vertex from $N[u]$ was chosen up to this point of the game. The game ends when no more legal moves can be made, and its length equals the number of vertices selected. The goal of Dominator is to minimize whilst that of Staller is to maximize the length of the game. The game domination number $\gamma_g(G)$ of G is the length of the domination game in which Dominator starts and both players play optimally. In this paper we establish an upper bound on $\gamma_g(G)$ in terms of the minimum degree δ and the order n of G . Our main result states that for every $\delta \geq 4$,

$$\gamma_g(G) \leq \frac{30\delta^4 - 56\delta^3 - 258\delta^2 + 708\delta - 432}{90\delta^4 - 390\delta^3 + 348\delta^2 + 348\delta - 432} n.$$

Particularly, $\gamma_g(G) < 0.5139 n$ holds for every graph of minimum degree 4, and $\gamma_g(G) < 0.4803 n$ if the minimum degree is greater than 4. Additionally, we prove that $\gamma_g(G) < 0.5574 n$ if $\delta = 3$.

Keywords: domination game, game domination number, 3/5-conjecture, minimum degree.

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1 Introduction

In this note, our subject is the domination game introduced by Brešar, Klavžar and Rall in [4].

1.1 Basic definitions

For a simple undirected graph $G = (V, E)$ and for a vertex $v \in V$, the *open neighborhood* of v is $N_G(v) = \{u : uv \in E\}$, while its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. Then the *degree* $d_G(v)$ of v is just $|N_G(v)|$ and the minimum degree $\min\{d_G(v) : v \in V\}$ is denoted by $\delta(G)$. As usual, we will write $N(v)$, $N[v]$ and $d(v)$ for $N_G(v)$, $N_G[v]$ and $d_G(v)$, respectively, if G is clear from the context.

Each vertex dominates itself and its neighbors, moreover a set $S \subseteq V$ dominates exactly those vertices which are contained in $N[S] = \bigcup_{v \in S} N[v]$. A vertex set $D \subseteq V$ is called a *dominating set* of G if $N[D] = V$. The smallest cardinality of a dominating set is the *domination number* $\gamma(G)$ of G .

The *domination game*, introduced by Brešar, Klavžar and Rall [4], is played on a simple undirected graph $G = (V, E)$ by two players, named Dominator and Staller, respectively. They take turns choosing a vertex from V such that a vertex v can be chosen only if it dominates at least one new vertex – that is, if we have a $u \in N[v]$ for which no vertex from $N[u]$ was selected up to this turn of the game. The game is over when no more legal moves can be made; equivalently, when the set D of vertices chosen by the two players becomes a dominating set of G . The aim of Dominator is to finish the game as soon as possible, while that of Staller is to delay the end of the game. The *game domination number* $\gamma_g(G)$ is the number of turns in the game when the first turn is Dominator's move and both players play optimally. Analogously, the *Staller-start game domination number* $\gamma'_g(G)$ is the length of the game when Staller begins and the players play optimally.

1.2 Results

Although the subject is quite new, lots of interesting results have been obtained on the domination game (see [2, 3, 4, 5, 6, 7, 9, 13, 14]). Note that also the total version of the domination game was introduced [11] and studied [12] recently.

Concerning our present work, the bounds proved for the game domination number $\gamma_g(G)$ are the most important preliminaries. The following fact was verified in [4] and [13] as well.

$$\gamma(G) \leq \gamma_g(G) \leq 2\gamma(G) - 1 \quad (1)$$

Upper bounds in terms of the order were inspired by the following “3/5-conjecture” raised by Kinnersley, West and Zamani [13].

Conjecture 1 *If G is an isolate-free graph of order n , then $\gamma_g(G) \leq 3n/5$ holds.*

Conjecture 1 has been proved for the following graph classes:

- for trees of order $n \leq 20$ (Brešar, Klavžar, Košmrlj and Rall [3]);
- for caterpillars – that is, for trees in which the non-leaf vertices induce a path (Kinnarsley, West and Zamani [13]);
- for trees in which no two leaves are at distance four apart (Bujtás [6, 7]).

Moreover, in a manuscript in preparation, Henning and Kinnarsley prove Conjecture 1 for graphs of minimum degree at least 2 [10].

On the other hand, upper bounds weaker than $3n/5$ were obtained for some wider graph classes. For trees, the inequality $\gamma_g(G) \leq 7n/11$ was established by Kinnarsley, West and Zamani in [13] and it was recently improved to $\gamma_g(G) \leq 5n/8$ by the present author in [7]. For the most general case, Kinnarsley, West and Zamani proved [13] that the game domination number of any isolate-free graph G of order n satisfies $\gamma_g(G) \leq \lceil 7n/10 \rceil$. In Section 2 we improve this upper bound by establishing the following claim.

Proposition 1 *For any isolate-free graph G of order n ,*

$$\gamma_g(G) \leq \frac{2n}{3} \quad \text{and} \quad \gamma'_g(G) \leq \frac{2n}{3}.$$

In fact, in a manuscript under preparation [8] we will prove the stronger inequality $\gamma_g(G) \leq 0.64n$, but the proof of Proposition 1 may be of interest because of its simplicity and gives illustration for the proof technique applied in the later sections.

One of our main results gives an upper bound smaller than $0.5574n$ on the game domination number of graphs with minimum degree 3.

Theorem 1 *For any graph G of order n and with minimum degree 3,*

$$\gamma_g(G) \leq \frac{34n}{61} \quad \text{and} \quad \gamma'_g(G) \leq \frac{34n - 27}{61}.$$

For graphs all of whose vertices are of degree greater than 3, we prove an upper bound in terms of the order and the minimum degree.

Theorem 2 *If G is a graph on n vertices and its minimum degree is $\delta(G) \geq d \geq 4$, then*

$$\gamma_g(G) \leq \frac{30d^4 - 56d^3 - 258d^2 + 708d - 432}{90d^4 - 390d^3 + 348d^2 + 348d - 432} n.$$

As the coefficient in this upper bound equals $37/72 < 0.5139$ for $d = 4$, and equals $2102/4377 < 0.4803$ for $d = 5$, the following immediate consequences are obtained.

Corollary 1

- (i) *For any graph G of order n and with minimum degree $\delta(G) = 4$, the inequality $\gamma_g(G) \leq 37n/72$ holds.*
- (ii) *For any graph G of order n and with minimum degree $\delta(G) \geq 5$, the inequality $\gamma_g(G) \leq 2102n/4377$ holds.*

Particularly, these statements show that the coefficient $3/5$ in Conjecture 1 can be significantly improved if only those graphs with $\delta(G) \geq 4$ are considered.

On the other hand, note that Theorem 1 and Theorem 2 establish new results only for $3 \leq \delta(G) \leq 21$. Although it was not mentioned in the earlier papers, the upper bound in (1) together with the well-known theorem (see e.g., [1])

$$\gamma(G) \leq \frac{1 + \ln(\delta + 1)}{\delta + 1} n$$

clearly yields

$$\gamma_g(G) < 2 \cdot \frac{1 + \ln(\delta + 1)}{\delta + 1} n \tag{2}$$

for each $\delta \geq 2$. For integers $3 \leq \delta(G) = d \leq 21$, it is easy to check that our bound is better than the above one in (2).

Our proof technique is based on a value assignment to the vertices where the value of a vertex depends on its current status in the game. We will consider a greedy strategy of Dominator, where the greediness is meant concerning the decrease in the values. Our main goal is to estimate the average decrease in a turn achieved under this assumption. We have been introduced this type of approach in the conference paper [6] and in the paper [7]. The frame of this technique and the basic observations are contained here in Section 2. Then, in Section 3 and Section 4 we specify the details and prove our Theorem 1 and Theorem 2 respectively. In the last section we make some additional notes concerning the Staller-start version of the game.

2 Preliminaries

Here we introduce the notion of the residual graph, define the color assignment to the vertices and give a general determination for the phases of the game. Then, we take some simple observations which will be used in the later sections.

Colors Consider any moment of the process of a domination game on the graph $G^* = (V, E)$, and denote by D the set of vertices chosen up to this point of the game. As it was introduced in [6] and [7], we distinguish between the following three types of vertices.

- A vertex $v \in V$ is *white* if $v \notin N[D]$.
- A vertex $v \in V$ is *blue* if $v \in N[D]$ but $N[v] \not\subseteq N[D]$.
- A vertex $v \in V$ is *red* if $N[v] \subseteq N[D]$.

Residual graph Clearly, a red vertex v and all its neighbors are already dominated in the game. Hence the choice of v would not be a legal move in the later turns and further, the status of v remains red. So, red vertices do not influence the continuation of the game and they can be deleted. Similarly, edges connecting two blue vertices can be omitted too. This graph, obtained after the deletion of red vertices and edges between two blue vertices, is called *residual graph*, as it was introduced in [13]. At any point of the game, the set of vertices chosen up to this point is denoted by D and the residual graph is denoted by G . When it is needed, we use the more precise notations D_i and G_i for the current D and G just before the i th turn.

Phases of the game The phases will be defined for the Dominator-start game that is, for each odd integer j the j th turn belongs to Dominator. The Staller-start version will be treated later by introducing a Phase 0 for the starting turn.

In our proofs, nonnegative values $p(v)$ are assigned to the vertices, and the value $p(G)$ of the residual graph is just the sum of the values of the vertices. Also, we assume that Dominator always chooses greedily. More precisely, for each odd j , in the j th turn he plays a vertex which results the possible maximum $p(G_j) - p(G_{j+1})$. This difference is called the decrease in the value of $p(G)$ and also referred to as the gain of the player.

Definition 1 Let $(C1), \dots, (C\ell)$ be conditions all of which relating to the j th turn of the game where j is odd. Then, for each $i = 1, \dots, \ell$, Phase i of the game is defined as follows.

- (i) Phase 1 begins with the first turn of the game.
- (ii) If Phase i begins with the b_i th turn, it is continued as long as (Ci) is satisfied in each turn of Dominator. That is, Phase i ends right after the e_i th turn where e_i is the smallest even integer with $b_i < e_i$ for which (Ci) is not satisfied in the $(e_i + 1)$ st turn.
- (iii) If Phase i ends after the e_i th turn but the game is not over yet, then the $(e_i + 1)$ st turn is the beginning of Phase i' , where i' is the smallest integer with $i < i'$ such that (Ci') is fulfilled in the $(e_i + 1)$ st turn.

- (iv) If Phase i is followed by Phase i' and $i + 2 \leq i'$ holds, we say that Phases $i + 1, \dots, i' - 1$ are skipped; moreover, their starting and end points are interpreted to be the same as the end of Phase i .

Further notations The colors white, blue and red will be often abbreviated to W, B and R, respectively. For example, a B-neighbor is a blue neighbor, and the notation $v: W \rightarrow B/R$ means that vertex v changed from white to either blue or red in the turn considered. Moreover, $d^W(v)$ and $d^B(v)$ stand for the number of W-neighbors and B-neighbors of v , respectively.

We cite the following observations (in a slightly modified form) from [7]:

Lemma 1 *The following statements are true for every residual graph G in a domination game started on G^* .*

- (i) *If v is a white vertex in G , then v has the same neighborhood in G as it had in G^* . Thus, $d_G(v) = d_{G^*}(v)$ holds for every W-vertex of G and moreover, $d_G^W(v) + d_G^B(v) = d_{G^*}(v)$.*
- (ii) *If v is a blue vertex in G , then v has only white neighbors and definitely has at least one. That is, $d_G^W(v) = d_G(v) \geq 1$ and $d_G^B(v) = 0$ if v is a B-vertex in G .*

At the end of this section, we provide a simple example for applying the tools introduced above. We prove Proposition 1, which states that for any isolate-free graph G of order n ,

$$\gamma_g(G) \leq \frac{2n}{3} \quad \text{and} \quad \gamma'_g(G) \leq \frac{2n}{3}$$

hold.

Proof of Proposition 1. First, we consider the Dominator-start game on $G^* = (V, E)$, which is a simple graph without isolated vertices. In every residual graph G , let the value $p(v)$ of a vertex v be equal to 2, 1 and 0, when v is white, blue and red, respectively. Hence, we start with $p(G^*) = 2n$ and assume that Dominator always selects a vertex which results in a maximum decrease in $p(G)$. The game is divided into two phases, which are determined due to Definition 1 with the following conditions:

- (C1) Dominator gets at least 4 points.
- (C2) Dominator gets at least 1 point.

Phase 1. If Staller selects a W-vertex, then it becomes red and causes at least 2-point decrease in the value of the residual graph. In the other case, Staller selects a B-vertex v which has a W-neighbor u . Then, the changes v : B→R and u : W→B/R together result in a decrease of at least $1 + 1 = 2$. Hence, in each of his turns Staller gets at least 2 points. By condition (C1), Dominator always gets at least 4 points. As Dominator begins the phase, the average decrease in $p(G)$ must be at least 3 in a turn.

Phase 2. When Phase 2 starts, Dominator cannot seize 4 or more points by playing any vertex of G_j . This implies the following properties of the residual graph:

- For every W-vertex v , $d^W(v) \leq 1$.
Indeed, if v had two W-neighbors u_1 and u_2 , then Dominator could choose v and the changes v : W→R and u_1, u_2 : W→B/R would give a gain of at least $2 + 2 \cdot 1 = 4$ points, which is a contradiction.
- For every W-vertex v , $d^W(v) = 0$.
We have seen that $d^W(v) \leq 1$. Now, assuming two W-neighbors v and u , the choice of v would result in the changes v, u : W→R, which give a gain of at least 4 points to the player. This is a contradiction again.
- For every B-vertex v , $d(v) = 1$.
By Lemma 1(ii), $d(v) \geq 1$. Now, assume that v has two different W-neighbors u_1 and u_2 . As we have shown, $d^W(u_1) = d^W(u_2) = 0$ must hold and consequently, if Dominator plays v , then both u_1 and u_2 turn to red. This gives a gain of at least $1 + 2 \cdot 2 = 5$ points, which cannot be the case at the endpoint of Phase 1.
- Each component of G_j is a P_2 and contains exactly one white and one blue vertex.
By the claims above, each component is a star with a white center and blue leaves. If it contained at least two leaves then Dominator could play the center and get at least 4 points.

Therefore, at the beginning of Phase 2 the residual graph consists of components of order 2. As follows, in each turn an entire component becomes red and $p(G)$ decreases by exactly 3 points.

In the game, the value of the residual graph decreased from $2n$ to zero, and the average decrease in a turn was proved to be at least 3. Consequently, the number of turns required is not greater than $2n/3$, which proves $\gamma_g(G^*) \leq 2n/3$.

If Staller starts the game, his first move definitely decreases $p(G)$ by at least 3 points as there are no isolated vertices. Then, the game is continued as in the Dominator-start game, and the average decrease remains at least 3 points. Thus, $\gamma'_g(G^*) \leq 2n/3$ holds.

□

3 Graphs of minimum degree 3

In this section we prove the upper bound stated on the game domination number of graphs with minimum degree 3. Also, this proof serves as an introduction to the details of the idea applied in the next section to prove our main theorem.

Proof of Theorem 1. We consider a graph $G = (V, E)$ of minimum degree 3 and define the value assignments of types A1.1, A1.2 and A1.3 as they are given in Table 1.

Table 1: Value assignments used in the proof of Theorem 1

Abbreviation	Type of the vertex	Value in A1.1	Value in A1.2	Value in A1.3
W	white vertex	34	34	34
B ₃	blue vertex of degree at least 3	16	16	—
B ₂	blue vertex of degree 2	16	13	13
B ₁	blue vertex of degree 1	16	10	9
R	red vertex	0	0	0

Hence, the game starts with $p(G^*) = 34n$. First, assume that Dominator begins the game and determine Phases 1-4 due to Definition 1 with the following specified conditions:

- (C1) Dominator gets at least 88 points due to the assignment A1.1.
- (C2) Dominator gets at least 91 points due to the assignment A1.2.
- (C3) Dominator gets at least 84 points due to the assignment A1.3.
- (C4) Dominator gets at least 1 point due to the assignment A1.3.

Phase 1. Here, we apply the value assignment A1.1. In each of his turns, Staller either selects a white vertex and gets at least 34 points; or he plays a blue vertex v which has a white neighbor u and then the color changes v : $B \rightarrow R$ and u : $W \rightarrow B/R$ give at least $16+18=34$ points. By condition (C1), each move of Dominator yields a gain of at least 88 points. As Dominator begins the game, we have the following estimation on the average decrease of $p(G)$ in a turn.

Lemma 2 *In Phase 1, the average decrease of $p(G)$ in a turn is at least 61 points.*

At the end of Phase 1 we have some structural properties which remain valid in the continuation of the game.

Lemma 3 *After the end of Phase 1, throughout the game, each white vertex has at most 2 white neighbors, and each blue vertex has at most 3 white neighbors.*

By definition, at the end of the first phase Dominator has no possibility to seize 88 or more points by playing a vertex of the residual graph G . Now, assume that there exists a W-vertex v with three W-neighbors u_1, u_2 and u_3 in G . Then, Dominator could choose v and the color changes v : $W \rightarrow R$ and u_1, u_2, u_3 : $W \rightarrow B/R$ would decrease $p(G)$ by at least $34 + 3 \cdot 18 = 88$ points, which is a contradiction. Similarly, if there exists a B-vertex v with at least four W-neighbors, then Dominator could get at least $16 + 4 \cdot 18 = 88$ points by playing v , which is a contradiction again.

In the continuation of the game, new white vertices cannot arise, moreover a new blue vertex may arise only by the color change $W \rightarrow B$. This implies that the stated properties remain valid throughout all the later phases. \square

Phase 2. At the beginning of this phase we change to assignment A1.2. Clearly, the values of the vertices do not increase. As no blue vertex has a degree greater than 3, we observe that each B-vertex v has value $p(v) = 7 + 3d(v)$. Further, assignment A1.2 ensures that when a vertex v is played, the value of every blue vertex from $N[N[v]]$ is decreased.

Lemma 4 *The following statements are true in Phase 2.*

- (i) *If a W-vertex v with W-degree $d^W(v)$ is played, then $p(G)$ decreases by at least $43 + 24d^W(v)$ points.*
- (ii) *If a B-vertex v with degree $d(v)$ is played, then $p(G)$ decreases by at least $7 + 24d(u)$ points.*
- (iii) *In each turn $p(G)$ decreases by at least 31 points*

Proof When the degree of a B-vertex is decreased by x , its value decreases by at least $3x$, no matter whether the change is of type $B_i \rightarrow B_{i-x}$ or $B_x \rightarrow R$. Thus, if a vertex v is played, the sum of the values of B-vertices contained in $N[N[v]] \setminus \{v\}$ is decreased by at least

$$3 \sum_{u \in N[v]} d^B(u) \geq 3 \sum_{u \in N[v]} (3 - d^W(u))$$

if v is white, and by at least

$$3 \sum_{u \in N[v]} (d^B(u) - 1) \geq 3 \sum_{u \in N[v]} (2 - d^W(u))$$

if v is blue.

First, assume that the played vertex v is white, $d^W(v) = k$ and the W-neighbors of v are u_1, \dots, u_k . For each $1 \leq i \leq k$, the W-vertex u_i becomes either a B-vertex of degree at most $d^W(u_i) - 1$ or an R-vertex. As $1 \leq d^W(u_i) \leq 2$, $p(u_i)$ decreases by at least $34 - (7 + 3d^W(u_i) - 3) = 30 - 3d^W(u_i)$ in either case. Then, the decrease in $p(G)$ is not smaller than

$$34 + \sum_{i=1}^k (30 - 3d^W(u_i)) + 3(3 - k) + 3 \sum_{i=1}^k (3 - d^W(u_i)) = 43 + 36k - 6 \sum_{i=1}^k d^W(u_i) \geq 43 + 24k,$$

where $0 \leq k \leq 3$ must hold. This establishes statement (i).

In the other case, v is blue with $d(v) = k$ and its W-neighbors are u_1, \dots, u_k . As v has only white neighbors and definitely has at least one and no more than 3, $1 \leq k \leq 3$ holds; moreover, $0 \leq d^W(u_i) \leq 2$ is true for all $1 \leq i \leq k$. When v is played, u_i becomes red if $d^W(u_i) = 0$, otherwise it will be a blue vertex of degree at most $d^W(u_i)$. Therefore, the decrease in $p(u_i)$ is at least $34 - (7 + 3d^W(u_i)) = 27 - 3d^W(u_i)$ and that in $p(v)$ is exactly $7 + 3k$. Then, the sum of the decreases cannot be smaller than

$$7 + 3k + \sum_{i=1}^k (27 - 3d^W(u_i)) + 3 \sum_{i=1}^k (2 - d^W(u_i)) = 7 + 36k - 6 \sum_{i=1}^k d^W(u_i) \geq 7 + 24k$$

as stated in (ii).

To prove (iii), it suffices to consider the minimum of $43 + 24k$ in case (i), which is 43; and that of $7 + 24k$ in case (ii), which is 31 because of the condition $k \geq 1$. \square

By Lemma 4(iii), Staller gets at least 31 points, and by Condition (C2), Dominator gets at least 91 points in each of their turns. Hence, we have the following estimation.

Lemma 5 *In Phase 2, the average decrease of $p(G)$ in a turn is at least 61 points.*

As shown by the next lemma, the W-degrees are more strictly bounded from the end of Phase 2 than earlier.

Lemma 6 *After the end of Phase 2, throughout the game, each white vertex has at most 1 white neighbors, and each blue vertex has at most 2 white neighbors.*

Proof By condition (C2), at the end of Phase 2 Dominator can seize only less than 91 points by choosing any vertex of G . By Lemma 4(i), the selection of a W-vertex v with $d^W(v) = 2$ causes a decrease of at least $43 + 24 \cdot 2 = 91$ points in $p(G)$. Hence, each W-vertex has either zero or exactly one W-neighbor.

Now, assume that v is a B-vertex with three W-neighbors, say u_1, u_2 and u_3 . We have already seen that the inequalities $0 \leq d^W(u_i) \leq 1$ hold for $i = 1, 2, 3$. Then, as

it was shown in the proof of Lemma 4(ii), the choice of v would decrease $p(G)$ by at least

$$7 + 36 \cdot 3 - 6 \sum_{i=1}^3 d^W(u_i) \geq 97,$$

which is a contradiction. \square

Phase 3. The phase starts with changing the value assignment A1.2 to A1.3. By Lemma 6, there are no B-vertices of degree 3 or higher, moreover we observe that the change to A1.3 cannot cause increase in the value of G . Also, one can easily check that the value of a B-vertex decreases by at least $4x$ points, if it loses x W-neighbors in a turn.

Lemma 7 *The following statements are true in Phase 3.*

- (i) *If a W-vertex v is played, then $p(G)$ decreases by at least 84 points if $d^W(v) = 1$, and $p(G)$ decreases by at least 46 points if $d^W(v) = 0$.*
- (ii) *If a B-vertex v is played, then $p(G)$ decreases by at least 67 points if $d(v) = 2$, and $p(G)$ decreases by at least 38 points if $d(v) = 1$.*
- (iii) *In each turn $p(G)$ decreases by at least 38 points*

Proof (i) Consider a W-vertex v whose only W-neighbor is u . By Lemma 6, all the further neighbors of v and u are blue and there are at least four such neighbors. Hence, when v is played, the color changes v, u : $W \rightarrow R$ decrease $p(G)$ by 68 points, while the sum of the values of B-vertices contained in $N[\{v, u\}]$ decreases by at least $4 \cdot 4 = 16$. Hence, the gain of the player is at least 84 points. In the other case, when v has no W-neighbors, it has at least three B-neighbors. Then, the change v : $W \rightarrow R$ gives at least 34 points and additionally, the decrease in the degrees of the B-neighbors means at least 12 points. This proves that $p(G)$ decreases by at least 46 points.

(ii) If the played vertex v is blue and has exactly one white neighbor u , then the changes v : $B_1 \rightarrow R$ and u : $W \rightarrow B_1/R$ cause a decrease of at least $9 + 25 = 34$ points in $p(G)$. Additionally, u has at least one B-neighbor different from v , whose value is decreased by at least 4 points. Consequently, the total decrease is at least 38 points. Similarly, if v is blue and has two W-neighbors u_1 and u_2 , then the total decrease in $p(G)$ is at least $13 + 2 \cdot 25 + 2 \cdot 4 = 71$ points.

(iii) As the four cases above cover all possible moves which can be made in Phase 3, $p(G)$ is decreased by at least 38 points in each turn. \square

As consequences of Condition (C3) and Lemma 7(iii), Dominator gets at least 84 points and Staller gets at least 38 points in each of his turns. Hence, we have the desired average.

Lemma 8 *In Phase 3, the average decrease of $p(G)$ in a turn is at least 61 points.*

When Dominator cannot get at least 84 points in a turn, the structure of the residual graph must be very simple.

Lemma 9 *At the end of Phase 3, each component of the residual graph is a star of order $k \geq 4$ with a white center and $k - 1$ blue leaves.*

Proof Let G_i be the residual graph obtained at the end of Phase 3. Due to Lemma 7(i), the presence of a W-vertex v with $d^W(v) = 1$ provides an opportunity for Dominator to get at least 84 points. Then, the i th turn would belong to Phase 3, which is a contradiction. Consequently, in G_i each W-vertex has only B-neighbors.

Next, assume that we have a B-vertex v which has two W-neighbors u_1 and u_2 in G_i . As we have seen, in G_i $d^W(u_1) = d^W(u_2) = 0$ must hold and moreover, both u_1 and u_2 have at least two B-neighbors. Therefore, if v is selected by Dominator, the changes $v: B_2 \rightarrow R$ and $u_1, u_2: W \rightarrow R$ with the change in the values of B-neighbors, all together yield at least $13 + 2 \cdot 34 + 4 \cdot 4 = 97$ point decrease in $p(G_i)$, which is a contradiction again. Hence, each B-vertex has at most one W-neighbor.

Since each W-vertex v has the same degree in the residual graph G_i as it had in G^* , it has at least three B-neighbors in G_i . In addition, each B-vertex is a leaf in G_i . This implies that at the end of Phase 3 every component is a star with the structure stated. \square

Phase 4. By Lemma 9, Phase 4 begins with star-components containing a white center and at least three blue leaves. Then, in each turn a component becomes completely red, no matter whether a white or a blue vertex is played. Thus, each move decreases the value of G by at least $34 + 3 \cdot 9 = 61$ points.

Lemma 10 *In Phase 4, the average decrease of $p(G)$ in a turn is at least 61 points.*

By Lemmas 2, 5, 8 and 10, if Dominator starts the game and he plays the prescribed greedy strategy, then for the number t^* of turns

$$\gamma_g(G^*) \leq t^* \leq \frac{34}{61}n$$

holds.

Finally, for the Staller-start version of the game we define Phase 0, which contains only the first turn and the values are counted due to A1.1. Observe that Staller's any choice results in at least $34 + 3 \cdot 18 = 88$ point decrease in $p(G^*)$. Then, Phase 1 might be skipped if (C1) is not true for G_1 , but otherwise the game continues as in the Dominator-start version and our lemmas remain valid. Therefore, by the 27 point overplus arising in Phase 0, for $\gamma'_g(G^*)$ we obtain a slightly better bound,

$$\gamma'_g(G^*) \leq \frac{34n - 27}{61}.$$

This completes the proof of Theorem 1. \square

4 Graphs with minimum degree greater than 3

Here we prove Theorem 2 and Corollary 1.

Proof of Theorem 2. First, we consider the Dominator-start game on a graph $G^* = (V, E)$ of order n , whose minimum degree is $\delta(G^*) \geq d \geq 4$.

The proof and the game starts with the value assignment A2.1 to the vertices as shown in Table 2. Later, we use a more subtle distinction between the types of blue vertices due to assignments A2.2, A2.3 and A2.4 (see Table 2). We will see that the value $p(G)$ of the residual graph cannot increase when we change to an assignment with a higher index. We consider a graph $G^* = (V, E)$ with a minimum degree

$\delta(G) \geq d \geq 4$ and define the value assignments of types A2.1, A2.2, A2.3 and A2.4 as they are given in Table 2.

Table 2: Value assignments used in the proof of Theorem 2

Abbreviation	Type of the vertex	A2.1	A2.2	A2.3	A2.4
W	white vertex	a	a	a	a
B ₄	blue vertex of degree at least 4	b	b	—	—
B ₃	blue vertex of degree 3	b	$b - x_1$	$b - x_1$	—
B ₂	blue vertex of degree 2	b	$b - 2x_1$	$b - x_1 - x_2$	$b - x_1 - x_2$
B ₁	blue vertex of degree 1	b	$b - 3x_1$	$b - x_1 - 2x_2$	$b - x_1 - x_2 - x_3$
R	red vertex	0	0	0	0

The values of a , b , x_1 , x_2 , x_3 and s are defined in terms of the parameter d . We aim to prove that s is a lower bound on the average decrease of $p(G)$ in a turn, if Dominator follows the prescribed greedy strategy.

$$\begin{aligned}
 a &= 30d^4 - 56d^3 - 258d^2 + 708d - 432 \\
 b &= 111d^3 - 561d^2 + 888d - 432 \\
 x_1 &= 6d^3 - 19d^2 + 15d \\
 x_2 &= 15d^3 - 64d^2 + 65d \\
 x_3 &= 30d^3 - 144d^2 + 202d - 72 \\
 s &= 90d^4 - 390d^3 + 348d^2 + 348d - 432
 \end{aligned}$$

Concerning the values above and the change between assignments, we take the following observations.

Lemma 11 *For every fixed integer $d \geq 4$:*

$$(i) \quad 0 < x_1 < x_2 < x_3 < b - x_1 - x_2 - x_3 < b < a \quad \text{and} \quad x_3 < a - b.$$

- (ii) For every $1 \leq i < j \leq 4$ and every residual graph G , $p(G)$ does not increase if the value assignment A2.i is changed to A2.j (assuming that A2.j is defined for G).

Proof The proof of (i) is based on a simple counting and estimation. Table 3 shows the differences and their exact values for $d = 4, 5, 6$. The comparison of coefficients verifies our statements for $d \geq 7$.

Table 3: Values of the differences for the proof of Lemma 11

		$d = 4$	$d = 5$	$d = 6$
x_1	$6d^3 - 19d^2 + 15d$	140	8408	21312
$x_2 - x_1$	$9d^3 - 45d^2 + 50d - 72$	56	250	624
$x_3 - x_2$	$15d^3 - 80d^2 + 137d - 288$	156	488	1110
$b - x_1 - x_2 - 2x_3$	$30d^3 - 190d^2 + 404d$	208	732	1776
$a - b$	$30d^4 - 167d^3 + 303d^2 - 180d$	1120	4550	12636
$a - b - x_3$	$30d^4 - 197d^3 + 447d^2 - 382d + 72$	768	3462	10200

Once (i) is proved, Table 2 shows that no vertex has greater value by A2.j than by A2.i, whenever $j > i$ holds. \square

Note that later we will use further relations between a, b, x_1, x_2, x_3 and s but these are equations, which can be verified by simple counting, so the details will be omitted.

The game is divided into five phases due to Definition 1 with the following five conditions:

- (C1) Dominator gets at least $5a - 4b$ points due to the assignment A2.1.
- (C2) Dominator gets at least $4a - 3b + (4d - 6)x_1$ points due to the assignment A2.2.
- (C3) Dominator gets at least $3a - 2b + 2x_1 + (3d - 2)x_2$ points due to the assignment A2.3.
- (C4) Dominator gets at least $2a + (2d - 2)x_3$ points due to the assignment A2.4.
- (C5) Dominator gets at least 1 point due to the assignment A2.4.

Thus, the game starts on $G^* = G_1$ with $p(G_1) = a \cdot n$, and ends with a residual graph whose value equals 0. Recall that Dominator plays a purely greedy strategy. Our goal is to prove that the average decrease in $p(G)$ is at least s points in a turn.

Phase 1 In each turn, the player either selects a W-vertex which turns red and hence $p(G)$ decreases by at least a points; or he selects a B-vertex v which has a W-neighbor u . In the latter case the changes $v: B \rightarrow R$ and $u: W \rightarrow B/R$ together yield a decrease of at least $b + (a - b) = a$ points. Therefore, Staller gets at least a points in each of his turns in Phase 1. By condition (C1), Dominator seizes at least $5a - 4b$ points and therefore, in any two consecutive turns $p(G)$ decreases by at least $6a - 4b = 2s$ points. As Dominator starts, the following statement follows.

Lemma 12 *In Phase 1, the average decrease of $p(G)$ in a turn is at least s points.*

Concerning the structure of the residual graph obtained at the end of this phase, we prove the following properties.

Lemma 13 *At the end of Phase 1,*

- (i) *If v is a W-vertex, then $d^W(v) \leq 3$.*
- (ii) *If v is a B-vertex, then $d(v) \leq 4$.*

Proof At the end of the phase, we have a residual graph G_i in which Dominator cannot get $5a - 4b$ or more points. Assuming a W-vertex v with W-neighbors u_1, u_2, u_3 and u_4 , Dominator could play v and the changes $v: W \rightarrow R$ and $u_1, u_2, u_3, u_4: W \rightarrow B/R$ would result in a decrease of at least $a + 4(a - b) = 5a - 4b$ points, which is a contradiction. In the other case, the choice of a B-vertex which has five W-neighbors would yield a gain of at least $b + 5(a - b) = 5a - 4b$ points, which is a contradiction again. \square

Phase 2 In this phase we apply the value assignment A2.2. By Lemma 13(ii), each B-vertex has degree smaller than or equal to 4. Moreover by the definition of A2.2 and by Lemma 11, in the j th turn the value of a B-vertex u decreases by at least $(d_{G_j}(u) - d_{G_{j+1}}(u))x_1$ points.

Lemma 14 *In Phase 2, the average decrease of $p(G)$ in a turn is at least s points.*

Proof If a W-vertex v is played, each of its neighbors has a decrease of at least x_1 points in its value, no matter whether this change on the neighbor is $B_i \rightarrow B_{i-1}$ or $B_1 \rightarrow R$ or $W \rightarrow B_i/R$. Then, playing a W-vertex results in at least $a + d \cdot x_1$ point decrease in $p(G)$.

In the other case, when the played vertex v is blue, the decrease in its value is at least $b - 3x_1$. As v has a W-neighbor u , whose W-degree is at most 3, the change $u: W \rightarrow B_i/R$ ($i \leq 3$) yields further at least $a - (b - x_1)$ points gain; and since u has at least $d - 4$ B-neighbors different from v , the total decrease in $p(G)$ is at least $(b - 3x_1) + a - (b - x_1) + (d - 4)x_1 = a + (d - 6)x_1$. This yields that Staller gets at least $a + (d - 6)x_1$ points whenever a white or a blue vertex is played by him.

Complying with (C2), each move of Dominator results in a gain of at least $4a - 3b + (4d - 6)x_1$ and consequently, in any two consecutive turns of Phase 2, $p(G)$ decreases by at least $5a - 3b + (5d - 12)x_1 = 2s$ points. This proves the lemma. \square

Lemma 15 *At the end of Phase 2,*

(i) *If v is a W-vertex, then $d^W(v) \leq 2$.*

(ii) *If v is a B-vertex, then $d(v) \leq 3$.*

Proof To prove (i), assume that Dominator selects a W-vertex v with W-neighbors u_1, u_2 and u_3 . Remark that each u_ℓ may have at most two W-neighbors different from v . Therefore, the changes $v: W \rightarrow R$ and $u_1, u_2, u_3: W \rightarrow B_i/R$ ($i \leq 2$) give at least $a + 3(a - b + 2x_1)$ points to Dominator. In addition, each of v, u_1, u_2 and u_3 has at least $d - 3$ B-neighbors. Hence the total decrease in $p(G)$ is at least $a + 3(a - b + 2x_1) + 4(d - 3)x_1 = 4a - 3b + (4d - 6)x_1$ points. In this case, Dominator's turn would belong to Phase 2. Hence for every W-vertex v , $d^W(v) \leq 2$ must hold at the end of Phase 2.

Part (ii) can be shown in a similar way but here we can refer to the property (i) proved above. The selection of a B-vertex v which has four W-neighbors, say u_1, u_2, u_3 and u_4 , would cause the color changes $v: B_4 \rightarrow R$ and $u_1, u_2, u_3, u_4: W \rightarrow B_i/R$ (where $i \leq 2$, due to part (i)). Moreover each u_j has at least $d - 3$ B-neighbors different from v . These would give a gain of at least

$$b + 4(a - b + 2x_1) + 4(d - 3)x_1 = 4a - 3b + (4d - 4)x_1 > 4a - 3b + (4d - 6)x_1$$

point to Dominator, which is impossible at the end of Phase 2. This verifies part (ii). \square

Phase 3 Here we apply the value assignment A2.3. By Lemma 15(ii), each B-vertex v has degree $d(v) \leq 3$ and hence, A2.3 is defined for all vertices of the residual graph. We observe concerning this phase that whenever the degree of a B-vertex v is reduced by y , its value decreases by at least yx_2 points.

Lemma 16 *In Phase 3, the average decrease of $p(G)$ in a turn is at least s points.*

Proof If Staller plays a W-vertex, he gets at least $a + dx_2$ points. In the other case, he plays a B-vertex v which has a W-neighbor u . By Lemma 15, $d^W(u) \leq 2$ and hence, u has at least $d - 3$ B-neighbors different from v . The changes $v: B_i \rightarrow R$ and $u: W \rightarrow B_i/R$ (where $i \leq 2$), together with the changes on the further B-neighbors of u , yields a decrease of at least

$$(b - x_1 - 2x_2) + a - (b - x_1 - x_2) + (d - 3)x_2 = a + (d - 4)x_2$$

in $p(G)$. Therefore, Staller gets at least $a + (d - 4)x_2$ points in each of his turns. By condition (C2), we have a lower bound on the gain of Dominator as well. These yield the sum

$$4a - 2b + 2x_1 + (4d - 6)x_2 = 2s$$

for any two consecutive turns of Phase 3, and we can conclude that the average is at least s indeed. \square

Lemma 17 *At the end of Phase 3,*

(i) *If v is a W-vertex, then $d^W(v) \leq 1$.*

(ii) *If v is a B-vertex, then $d(v) \leq 2$.*

Proof At the end of Phase 3 we have a residual graph G_i , in which the choice of any vertex decreases $p(G_i)$ by strictly less than $3a - 2b + 2x_1 + (3d - 2)x_2$ points.

(i) Playing a W-vertex v , which has two W-neighbors say u_1 and u_2 , results in the changes v : $W \rightarrow R$ and u_1, u_2 : $W \rightarrow B_1/R$; additionally $d^B(v) + d^B(u_1) + d^B(u_2) \geq 3(d - 2)$. This means a decrease of at least

$$a + 2(a - b + x_1 + 2x_2) + 3(d - 2)x_2 = 3a - 2b + 2x_1 + (3d - 2)x_2$$

in $p(G_i)$. This cannot be the case; so each W-vertex has either zero or exactly one W-neighbor in G_i .

(ii) Now suppose that a B-vertex v with W-neighbors u_1, u_2 and u_3 is played in G_i . We have already seen that $d^W(u_j) \leq 1$ holds for every W-vertex u_j in G_i . Then, we have the changes v : $B_3 \rightarrow R$ and u_1, u_2, u_3 : $W \rightarrow B_1/R$. Further, each vertex from $\{u_1, u_2, u_3\}$ has at least $d - 2$ B-neighbors different from v . Hence, the total gain of the player would be at least

$$b - x_1 + 3(a - b + x_1 + 2x_2) + 3(d - 2)x_2 > 3a - 2b + 2x_1 + (3d - 2)x_2.$$

This contradiction proves (ii). □

Phase 4 First, we change to assignment A2.3. By Lemma 17(ii), in any residual graph of Phase 4, the W-vertices induce a subgraph consisting of isolated vertices and P_2 -components; moreover, each blue vertex has at most 2 (white) neighbors. Moreover, by Table 2 and Lemma 3, if a B-vertex loses y W-neighbors in a turn, its value is reduced by at least yx_3 points.

Lemma 18 *In Phase 4, the average decrease of $p(G)$ in a turn is at least s points.*

Proof If Staller selects a W-vertex v , each neighbor u of v has a decrease of at least x_3 in its value. Hence, the total decrease in $p(G)$ is not smaller than $b - x_1 - x_2 + dx_3$.

If Staller selects a B-vertex v , the change is either v : $B_2 \rightarrow R$ or v : $B_1 \rightarrow R$, it means at least $(b - x_1 - x_2 - x_3)$ -point gain. As $d(v) \geq 1$, we necessarily have a W-neighbor u of v whose change is u : $W \rightarrow B_1/R$. Further, u has at least $d - 2$ B-neighbors different from v . Therefore, the decrease in $p(G)$ is at least

$$(b - x_1 - x_2 - x_3) + a - (b - x_1 - x_2 - x_3) + (d - 2)x_3 = a + (d - 2)x_3.$$

Hence, in any case, Staller gets at least $a + (d - 2)x_3$ points in a turn of his own. By (C4), Dominator gets at least $2a + (2d - 2)x_3$ point in each of his turns and as follows, the average gain is at least

$$\frac{1}{2}(a + (d - 2)x_3 + 2a + (2d - 2)x_3) = s$$

points as stated. \square

Lemma 19 *At the end of Phase 4,*

- (i) *Every W-vertex has only B-neighbors.*
- (ii) *Every B-vertex has exactly one W-neighbor.*

Proof Consider G_i which is the residual graph obtained at the end of Phase 4. As (C4) is not true, Dominator cannot get $2a + (2d - 2)x_3$ or more points in the i th turn. By Lemma 17, if (i) is not true, we have a "white-pair" (v, u) , where u is the only W-neighbor of v and vice versa. Then, the choice of v would result the changes v, u : W \rightarrow R. This, together with the fact $d^B(v) + d^B(u) \geq 2d - 2$, implies that the decrease in $p(G_i)$ is at least $2a + (2d - 2)x_3$, which is a contradiction. Thus, (i) is true.

To prove (ii) we suppose for a contradiction that a B-vertex v has two W-neighbors u_1 and u_2 . By (i), these neighbors are "single-white" vertices and they turn to red if v is played; in addition both u_1 and u_2 has at least $d - 1$ B-neighbors different from v . Hence, selecting v Dominator could seize at least

$$(b - x_1 - x_2) + 2a + 2(d - 1)x_3 > 2a + (2d - 2)x_3$$

points, which is a contradiction again. \square

Phase 5 By Lemma 19(ii), the residual graphs occurring in this phase have simple structure, each of their components is a star of order at least $d + 1$ whose center is white and the leaves are blue. Then, in each turn of Phase 5 exactly one such a star component becomes completely red, no matter whether a white or a blue vertex is played. Then, the value of the residual graph is decreased by at least $a + d(b - x_1 - x_2 - x_3) = s$ points in each single turn.

Lemma 20 *In Phase 5, the average decrease of $p(G)$ in a turn is at least s points.*

By Lemmas 12, 14, 16, 18 and 20, the average decrease per turn in the residual graph is at least s for the entire game. As $p(G_1) = an$ and the changes between assignments nowhere caused increase in $p(G)$, the domination game where Dominator plays the described greedy strategy yields a game with at most an/s turns. This establishes Theorem 2. \square

5 Concluding remarks on the Staller-start game

In our main theorem, we do not give upper bound on $\gamma'_g(G)$ for graphs with $\delta(G) \geq d \geq 4$. It is quite clear from the proof that we can establish the same upper bound on $\gamma'_g(G)$ as proved for $\gamma_g(G)$. Moreover, a slight improvement on it is also possible. We close the paper with this complicated formula.

If Staller begins the game, we index this starting turn by zero and take it into Phase 0. Then, from the first turn of Dominator, it continues as in the proof of Theorem 2. In the turn of Phase 0, Staller gets at least

$$a + d(a - b) = s + 30d^5 - 227d^4 + 637d^3 - 786d^2 + 360d$$

points. In later phases, the average decrease remains at least s . This proves that

$$\gamma'_g(G) \leq \frac{(30d^4 - 56d^3 - 258d^2 + 708d - 432)n - 30d^5 + 227d^4 - 637d^3 + 786d^2 - 360d}{90d^4 - 390d^3 + 348d^2 + 348d - 432}$$

holds for every $d \geq 4$ and for every graph G of minimum degree not smaller than d .

References

- [1] N. Alon and J. H. Spencer, *The Probabilistic Method*, Wiley, 3rd edition, 2008.
- [2] B. Brešar, P. Dorbec, S. Klavžar and G. Košmrlj, Domination game: effect of edge- and vertex-removal, *Discrete Mathematics*, **330** (2014), 1–10.
- [3] B. Brešar, S. Klavžar, G. Košmrlj and D. F. Rall, Domination game: extremal families of graphs for the 3/5-conjectures, *Discrete Applied Mathematics*, 161 (2013), 1308–1316.
- [4] B. Brešar, S. Klavžar and D. F. Rall, Domination game and an imagination strategy, *SIAM Journal on Discrete Mathematics*, **24** (2010), 979–991.
- [5] B. Brešar, S. Klavžar and D. F. Rall, Domination game played on trees and spanning subgraphs, *Discrete Mathematics*, **313** (2013), 915–923.
- [6] Cs. Bujtás, Domination game on trees without leaves at distance four, *Proceedings of the 8th Japanese-Hungarian Symposium on Discrete Mathematics and Its Applications* (A. Frank, A. Recski, G. Wiener, eds.), June 4–7, 2013, Veszprém, Hungary, (2013), 73–78.
- [7] Cs. Bujtás, Domination game on forests, manuscript (2014).
- [8] Cs. Bujtás, General upper bound on the game domination number, manuscript (2014).

- [9] P. Dorbec, G. Košmrlj and G. Renault, The domination game played on unions of graphs, manuscript (2014).
- [10] M. A. Henning, W. B. Kinnersley, Personal communication (2014).
- [11] M. A. Henning, S. Klavžar and D. F. Rall, Total version of the domination game, manuscript (2013).
- [12] M. A. Henning, S. Klavžar and D. F. Rall, Total Domination Game Played in Forests, manuscript (2014).
- [13] W. B. Kinnersley, D. B. West and R. Zamani, Extremal problems for game domination number *SIAM Journal on Discrete Mathematics*, **27** (4) (2013), 2090 –2107.
- [14] G. Košmrlj, Realizations of the game domination number, *Journal of Combinatorial Optimization* (2014), to appear.